Teacher Note

Describing, Comparing, and Classifying Addition Strategies

In this unit, students describe and classify their addition strategies (Investigation 2) and their subtraction strategies (Investigation 4) by focusing on the way they start to solve a problem—their first steps. Strategies are made public in this way so that all students can benefit from seeing the different addition and subtraction methods. Students are encouraged to expand their repertoire of strategies so that they continue to become more flexible and fluent in their computation. Comparing different solutions in the same category also offers the opportunity to discuss how to become more efficient in solving problems by adding or subtracting larger "chunks" of the numbers.

Strategies are classified by the first step students take in solving the problem. The first step generally indicates how students are thinking about the problem. For example, consider the following problem:

$$439 + 363 =$$

Jill starts by adding 400 + 300, and Enrique starts by adding 439 + 300. Jill is adding by place, starting with the largest place. Enrique is breaking the second number into parts and adding each part to the first number.

Identifying the strategies helps students understand the mathematics of their work. As you listen to students explain their strategies, model language they can use to describe their methods by reflecting back to them what they are doing. For example, you may say that, "I see you are breaking both numbers apart by place, starting with the hundreds," or "Are you breaking 363 into parts and adding each part to 439?" Your language can also help students notice similarities between variations of a method: "Steve is also breaking apart both numbers by place, but he added the tens first, then the hundreds, and then the ones." Ask students to compare their methods with those that have been shared: "Who else broke up 363 into parts and then added each part to 439? Ursula broke up the 363 differently than Enrique did—she broke it into 300 + 61 + 2 because she noticed that she could add 61 to 439 to get 500 as her first step."

Let students decide as a class which methods should be grouped together on one chart and which are different. This work is about helping students make sense of a variety of solutions; it is not about "matching" their work to predetermined categories. However, you may have to guide the discussion to keep the number of categories reasonable and useful. Variations of similar methods by different students—such as Enrique's and Ursula's—can go on the same chart.

Some students combine methods. For example, students may start by changing one of the numbers in the problem, then solve the problem by using another method, and then adjust for the change. For example, Lucy solved 439 + 363 this way:

$$440 + 300 = 740$$
$$740 + 63 = 803$$
$$803 - 1 = 802$$

She started by adding 1 to 439 to make it 440, added 363 in parts, and then adjusted for her initial change by subtracting 1. In the discussions of strategies, you may want to ask students who are not combining methods to share their methods first so that some clear categories can be established. Then students can decide how to classify a method like Lucy's. They may classify it according to its first step as "changing one number to make it easier, and then adjusting at the end;" they may make a new chart of "mixed methods;" or they may want to label the variations on the "changing one number" chart with the ways that each is continued. Students and adults who are fluent with computation often use a mixture of methods in the way that Lucy does.

Teacher Note

Reasoning and Proof in Mathematics

As students develop strategies to perform calculations, they frequently make claims about numerical relationships. Part of the work of fourth grade involves helping students to strengthen their ability to verbalize those claims and consider such questions as: Does this claim hold for *all* numbers? How can we know? Finding ways to answer these questions provides the basis for making sense of formal proof when it is introduced years from now. Consider the following vignette in which students in a fourth grade class are discussing their methods for solving addition problems.

Amelia: Here is what I did for 229 + 347, I changed the 229 to 230. Then I took the 1 away from the 347. That makes it 230 + 346. I can do that in my head, it's 576.

Teacher: You are saying that 229 + 347 = 230 + 346?

Amelia: Yes, you can take the 1 off one number and put it on the other. The answer is the same.

Teacher: Did anyone else do a problem this way?

Benson: I did something like that, but not exactly. I was working on 597 + 375. I turned it into 600 + 372 by moving 3 from the 375 to make 600. Then I know the answer is 972.

Teacher: So 597 + 375 = 600 + 372. How is what Benson did the same as what Amelia did and how is it different?

Ramona: They both made one number larger and one number smaller. Amelia used 1 and Benson used a 3.

Amelia: You will get the same answer. If you take some number from one and put it on the other, the answer has to stay the same.

Teacher: Amelia, you are saying something big. Are you saying that with *any* addition problem, you can change the addends by adding something to one and subtracting that same thing from the other and still get the same answer? Is that what you mean?

Amelia: Yes. You still have the same amount.

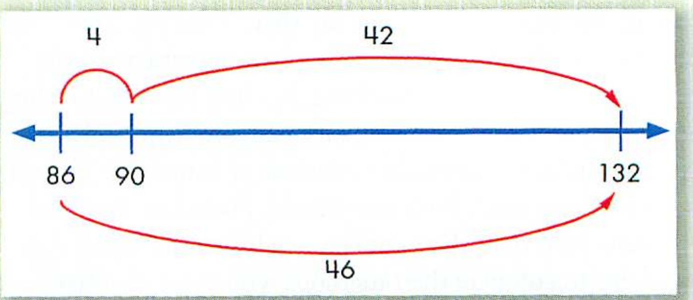
Teacher: Amelia has a way of thinking about this. I'd like all of us to work on this idea. How can you explain why this is true and how it works? You can use story contexts or diagrams or number lines, any of the tools we use to explain mathematics. You might want to begin by thinking about a particular problem like 86 + 46, but you need to show how your explanation works for *all* numbers, not just that example.

In this class, Amelia has made an assertion—mathematicians call such an assertion a conjecture—that if you add a certain amount to one of the addends and subtract it from the other addend, the sum remains the same. The teacher has challenged the class to find a way to show that this conjecture is true.

Let us return to the Grade 4 classroom to see how the students responded to the teacher's challenge to justify their conjecture.

Ramona: I was thinking of a story. Suppose I have 86 apples in one bag and 46 apples in another bag. I can take 4 of the apples out of the second bag and put them in the first bag. That means 86 + 46 = 90 + 42. You still have the same amount of apples.

Luke: I made a number line. You can see 86 + 46 is the same as 90 + 42. With mine it is like you add a piece of the 46 first and then add the rest of it.



Teacher: Ramona's story and Luke's number line both show us that 86 + 46 = 90 + 42. How can we use their work to say this works for all numbers?

Yuson: With the bags of apples, it doesn't matter how many are in each bag or how many you move. If you just take some out of one bag and put them in the other, it has to be the same amount.

Luke: It is the same with my number line. As long as the two numbers on the top add up to the one on the bottom, you have to land at the same place and that is the answer—no matter where you begin.

Ramona has used a story context that represents addition to explain how subtracting 4 from one addend and adding 4 to the other will result in the same total. Luke has drawn a number line to illustrate the same relationship. Yuson then explains how Ramona's story context will work with any addition problem with two addends. Similarly Luke explains how his number line model can apply to other addition problems.

Students in Grades K–5 can work productively on developing justifications for mathematical ideas as this class does here. But what is necessary to prove an idea in mathematics? First we'll examine what proving is in the field of mathematics, then we will return to the kind of proving students can do in fourth grade.

What Is Proof in Mathematics?

Throughout life, when people make a claim or assertion, they are often required to justify the claim, to persuade others that it is valid. A prosecutor who claims a person is guilty of a crime must make an argument, based on evidence, to convince the jury of this claim. A scientist who asserts that the earth's atmosphere is becoming warmer must marshal evidence, usually in the form of data and accepted theories and models, to justify the claim. Every field, including the law, science, and mathematics, has its own accepted standards and rules for how a claim must be justified in order to persuade others.

When K–5 students are asked to give reasons why *their* mathematical claims are true, they often say things like: "It worked for all the numbers we could think of." "I kept on trying and it kept on working." "We asked the 6th graders and they said it was true." "We asked our parents." These are appeals to particular instances and to authority. In any field, there are appropriate times to turn to authority (a teacher or a book, for example) for help with new knowledge or with an idea that we don't yet have enough experience to think through for ourselves. Similarly, particular examples can be very helpful in understanding some phenomenon. However, neither an authoritative statement nor a set of examples is sufficient to prove a mathematical assertion about an infinite class (say, all whole numbers).

In mathematics, a *theorem* must start with a mathematical assertion, which has explicit hypotheses (or "givens") and an explicit conclusion. The proof of the theorem must show how the conclusion follows logically from the hypotheses. A mathematical argument is based on logic and gives a sense of why a proposition is true. For instance, Ramona asserted that the sum of two addends remains the same if you subtract a certain number from one of the addends and add the same number to the other addend. In later years, Ramona's statement might be expressed as $(a + b) = (a + n) + (b - n)$. Luke's statement might be expressed as $a + (b + c) = (a + b) + c$. The proof of these claims consists of a series of steps in which one begins with the hypothesis— that a and b are numbers—and follows a chain of logical deductions ending with the conclusion— $(a + b) = (a + n) + (b - n)$ or $a + (b + c) = (a + b) + c$.

Each deduction must be justified by an accepted definition, fact, or principle, such as the commutative or associative property of addition. Luke's way of thinking about the problem is related to the associative property of addition. As Luke shows, the quantity 46 is equivalent to the sum 4 + 42, so 86 + 46 = 86 + (4 + 42). The associative property of addition indicates that in an addition expression such as 86 + (4 + 42), the addends can be regrouped without changing the value of the sum:

$a + (b + c) = (a + b) + c$. In this example, $86 + 46 = 86 + (4 + 42) = (86 + 4) + 42 = 90 + 42$.

The model for such a notion of proof was first established by Euclid, who codified what was known of Ancient Greek geometry in his *Elements*, written about 300 B.C. In his book, Euclid begins with the basic terms of geometry (a point, a line) and their properties (a line is determined by two points) and, through hundreds of propositions and proofs, moves to beautiful and surprising theorems about geometric figures.

What Does Proof Look Like in Fourth Grade?

One does not expect the rigor or sophistication of a formal proof, or the use of algebraic symbolism, from young children. Even for a mathematician, precise validation is often developed *after* new mathematical ideas have been explored and are more solidly understood. When mathematical ideas are evolving and there is a need to communicate the sense of *why* a claim is true, then informal methods of proving are appropriate. Such methods can include the use of visual displays, concrete materials, or words. The test of the effectiveness of such a justification is: Does it rely on logical thinking about the mathematical relationships rather than on the fact that one or a few specific examples work?

An important part of the fourth grader's justification is Yuson's statement that it doesn't matter what the numbers are. She understands that the story context Ramona uses as a model of addition can be used to show how subtracting an amount from one addend and adding the same amount to the other addend results in the same total, no matter what the original numbers are.

Proving by calling upon a model that represents the operation, as these students do by having mental images of the two addends and the act of putting them together, is particularly appropriate in K–5 classrooms where mathematical ideas are generally "under construction," and in which sense-making and diverse modes of reasoning are valued. The fourth graders' argument offers justification for the claim that if you subtract an amount from one addend and add that same amount to the other, the total remains the same. For Ramona, the sum of the numbers a and b is represented by the amount in the two bags. Moving some amount from one bag to the other does not change the total. Ramona's argument not only establishes the validity of the claim for particular numbers, but for any whole numbers, and easily conveys why it is true. Luke's number line diagram offers a slightly different version of the argument for the statement: If you break one of the addends into two parts and first add one and then add the other, the total remains the same.

To support the kind of reasoning illustrated in the vignette, teachers should encourage students to use representations such as cubes, story contexts, or number lines to explain their thinking. The use of representations offers a reference for the student who is explaining his or her reasoning, and it also allows more classmates to follow that reasoning. If it seems that students may be thinking only in terms of specific numbers, teachers might ask, Will that work for other numbers? How do you know? Will the explanation be the same?